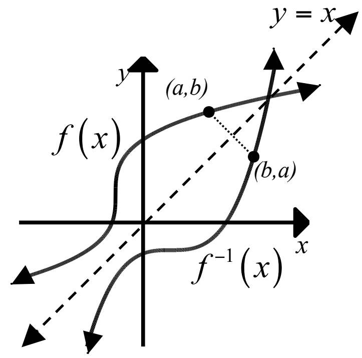
DERIVATIVES OF INVERSE FUNCTIONS



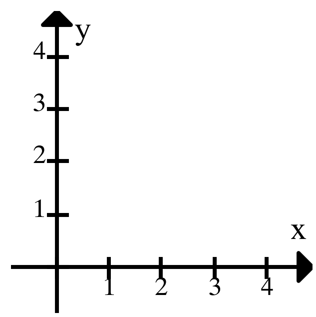
At right are the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$. Remember that if the graph of f contains the point (a, b), then the graph of f^{-1} contains the point (b, a). Also, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

What do you notice about the slope of the graph of f at (a, b) and the slope of the graph of f^{-1} at (b, a)?

Example 1: Let $f(x) = \sqrt{x}$.

a. Sketch the graph of $f(x)$.

b. Find $f^{-1}(x)$.



c. Sketch the graph of $f^{-1}(x)$ in the same coordinate plane as the graph of $f(x)$.

d. Differentiate both $f(x)$ and $f^{-1}(x)$.

e. Find the slope of the graph of $f(x)$ at (4, 2) and the slope of the graph of $f^{-1}(x)$ at (2, 4).

f. What conclusion can you make about these slopes?

Since slope $= m = \dfrac{\Delta y}{\Delta x}$, it should make sense that switching x and y (for inverse functions) should produce reciprocal slopes for inverse functions.

Derivatives of Inverse Functions:

If (a, b) is a point on f, then (b, a) is a point on f^{-1}, and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

or if f and g are inverse functions, then $g'(x) = \dfrac{1}{f'\left(g(x)\right)}$.

Derivatives of inverses have reciprocal slopes at "image points" (points reflected across
$y = x$). (a, b) and (b, a) are image points.

Let's derive the formula above:

Example 2: Let f and g be inverse functions such that:

$f(-1) = 1$	$f(0) = 2$	$f(1) = 5$
$f'(-1) = \frac{3}{2}$	$f'(0) = 2$	$f'(1) = \frac{1}{2}$

From the given information, find each of the following if possible.
Hint: Make a table or chart to organize your data.

a. $g'(1)$ b. $g'(2)$ c. $g'(3)$ d. $g'(0)$ e. $g'(5)$

Example #3: If $(2, 4)$ is a point on $f(x) = x^3 - 2x^2 + 2x$ then what is $\left(f^{-1}\right)'(4)$?