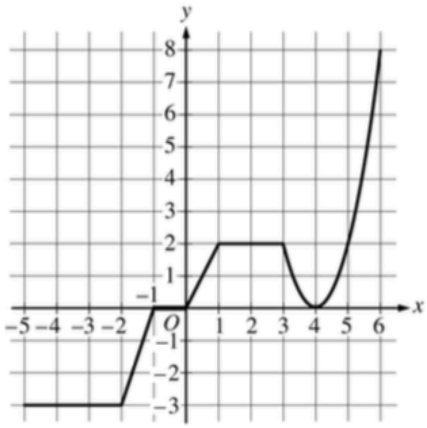
# AP Test Prep Questions, Week 4

All of the following questions are NonCalculator.

## 2018, #3




Graph of $g$

The graph of the continuous function $g$, the derivative of the function $f$, is shown above. The function $g$ is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$ ?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of $f$ both increasing and concave up? Give a reason for your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $f$. Give a reason for your answer.

## 2018, #4

| $t$ (years) | 2 | 3 | 5 | 7 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (meters) | 1.5 | 2 | 6 | 11 | 15 |

The height of a tree at time $t$ is given by a twice-differentiable function $H$, where $H(t)$ is measured in meters and $t$ is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time $t$, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate $\int_2^{10} H(t)dt$.

(d) The height of the tree, in meters, can also be modeled by the function $G$, given by $G(x) = \dfrac{100x}{1+x}$, where $x$ is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

**2018, #5**

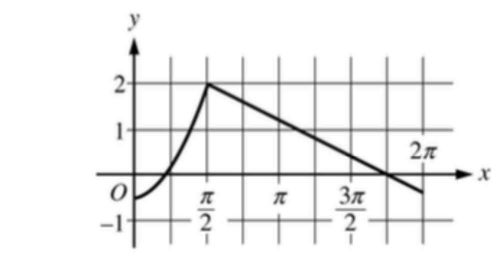
Let $f$ be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of $f$ on the interval $0 \le x \le \pi$.

(b) What is the slope of the line tangent to the graph of $f$ at $x = \frac{3\pi}{2}$ ?

(c) Find the absolute minimum value of $f$ on the interval $0 \le x \le 2\pi$. Justify your answer.

(d) Let $g$ be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of $g'$, the derivative of $g$, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.




Graph of $g'$

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**2017, #4**

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time $t$ minutes can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of $H$ at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time $t$ minutes is the function $G$ that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$ ?

**2010, Form B, #6**

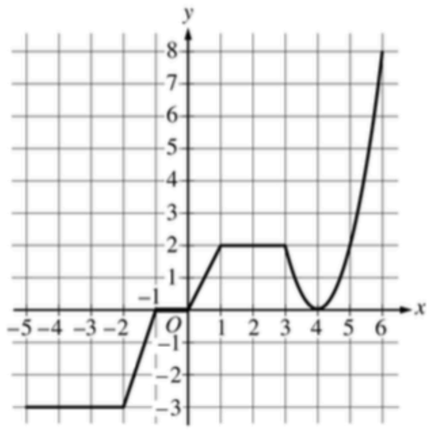
Two particles move along the $x$-axis. For $0 \le t \le 6$, the position of particle $P$ at time $t$ is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle $R$ at time $t$ is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

(a) For $0 \le t \le 6$, find all times $t$ during which particle $R$ is moving to the right.

(b) For $0 \le t \le 6$, find all times $t$ during which the two particles travel in opposite directions.

(c) Find the acceleration of particle $P$ at time $t = 3$. Is particle $P$ speeding up, slowing down, or doing neither at time $t = 3$? Explain your reasoning.

## 2018, #3




Graph of $g$

The graph of the continuous function $g$, the derivative of the function $f$, is shown above. The function $g$ is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$ ?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of $f$ both increasing and concave up? Give a reason for your answer.

(d) Find the $x$-coordinate of each point of inflection of the graph of $f$. Give a reason for your answer.

| | |
|---|---|
| (a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx$ <br> $= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$ | $2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$ |
| (b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$ <br> $= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx$ <br> $= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$ | $3: \begin{cases} 1: \text{split at } x = 3 \\ 1: \text{antiderivative of } 2(x-4)^2 \\ 1: \text{answer} \end{cases}$ |
| (c) The graph of $f$ is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals. | $2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$ |
| (d) The graph of $f$ has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$. | $2: \begin{cases} 1: \text{answer} \\ 1: \text{reason} \end{cases}$ |

**2018, #4**

| $t$ (years) | 2 | 3 | 5 | 7 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (meters) | 1.5 | 2 | 6 | 11 | 15 |

The height of a tree at time $t$ is given by a twice-differentiable function $H$, where $H(t)$ is measured in meters and $t$ is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time $t$, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate $\int_2^{10} H(t)dt$.

(d) The height of the tree, in meters, can also be modeled by the function $G$, given by $G(x) = \frac{100x}{1+x}$, where $x$ is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

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(a) $H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$

Because $H$ is differentiable on $3 \le t \le 5$, $H$ is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value $c$, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

(c)

$$\int_2^{10} H(t)\,dt \approx \left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3\right)$$

1 : trapezoidal sum

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1 + x)100 - 100x \cdot 1}{(1 + x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1 + x)^2} \cdot \frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{(1 + 1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0] if no chain rule

## 2018, #5

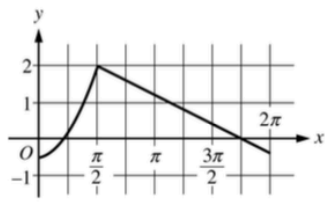
Let $f$ be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of $f$ on the interval $0 \le x \le \pi$.

(b) What is the slope of the line tangent to the graph of $f$ at $x = \frac{3\pi}{2}$?

(c) Find the absolute minimum value of $f$ on the interval $0 \le x \le 2\pi$. Justify your answer.

(d) Let $g$ be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of $g'$, the derivative of $g$, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.




Graph of $g'$

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(a) The average rate of change of $f$ on the interval $0 \le x \le \pi$ is $\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^{\pi} - 1}{\pi}$.

1 : answer

(b) $f'(x) = e^x \cos x - e^x \sin x$

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$$

The slope of the line tangent to the graph of $f$ at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$.

$2 : \begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$

(c) $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

| $x$ | $f(x)$ |
|---|---|
| $0$ | $1$ |
| $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}} e^{\pi/4}$ |
| $\frac{5\pi}{4}$ | $-\frac{1}{\sqrt{2}} e^{5\pi/4}$ |
| $2\pi$ | $e^{2\pi}$ |

The absolute minimum value of $f$ on $0 \le x \le 2\pi$ is $-\frac{1}{\sqrt{2}} e^{5\pi/4}$.

$3 : \begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4}, x = \frac{5\pi}{4} \\ \quad \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim_{x \to \pi/2} f(x) = 0$

Because $g$ is differentiable, $g$ is continuous.

$\lim_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$$\lim_{x \to \pi/2} \frac{f(x)}{g(x)} = \lim_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}.$$

$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal 0} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0-0] if no limit notation attached to a ratio of derivatives

## 2017, #4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time $t$ minutes can be modeled by the function $H$ that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

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(c) For $t < 10$, an alternate model for the internal temperature of the potato at time $t$ minutes is the function $G$ that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$ ?

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(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of $H$ is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

**2010, Form B, #6**

Two particles move along the $x$-axis. For $0 \le t \le 6$, the position of particle $P$ at time $t$ is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle $R$ at time $t$ is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

(a) For $0 \le t \le 6$, find all times $t$ during which particle $R$ is moving to the right.

(b) For $0 \le t \le 6$, find all times $t$ during which the two particles travel in opposite directions.

(c) Find the acceleration of particle $P$ at time $t = 3$. Is particle $P$ speeding up, slowing down, or doing neither at time $t = 3$? Explain your reasoning.

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| | |
|---|---|
| (a) $r'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$<br>$r'(t) = 0$ when $t = 1$ and $t = 3$<br>$r'(t) > 0$ for $0 < t < 1$ and $3 < t < 6$<br>$r'(t) < 0$ for $1 < t < 3$<br><br>Therefore $R$ is moving to the right for $0 < t < 1$ and $3 < t < 6$. | $2: \begin{cases} 1: r'(t) \\ 1: \text{answer} \end{cases}$ |
| (b) $p'(t) = -2 \cdot \frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right)$<br>$p'(t) = 0$ when $t = 0$ and $t = 4$<br><br>$p'(t) < 0$ for $0 < t < 4$<br>$p'(t) > 0$ for $4 < t < 6$<br><br>Therefore the particles travel in opposite directions for $0 < t < 1$ and $3 < t < 4$. | $3: \begin{cases} 1: p'(t) \\ 1: \text{sign analysis for } p'(t) \\ 1: \text{answer} \end{cases}$ |
| (c) $p''(t) = -2 \cdot \frac{\pi}{4} \cdot \frac{\pi}{4}\cos\left(\frac{\pi}{4}t\right)$<br>$p''(3) = -2\left(\frac{\pi}{4}\right)^2 \cos\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{8} \cdot \frac{\sqrt{2}}{2} > 0$<br>$p'(3) < 0$<br>Therefore particle $P$ is slowing down at time $t = 3$. | $2: \begin{cases} 1: p''(3) \\ 1: \text{answer with reason} \end{cases}$ |