AP Test Prep Questions, Week 3

All of the following questions are NonCalculator.

2006, Form B, #6

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ $\left(\text{ft/sec}^2\right)$	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Approximate $\int_{30}^{60} |v(t)|dt$ using a trapezoidal approximation with the three subintervals determined by the table.

(b) Find the exact value of $\int_0^{30} a(t)dt$.

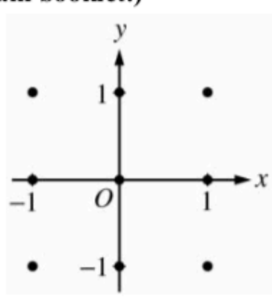
(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

2006, Form B, #5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

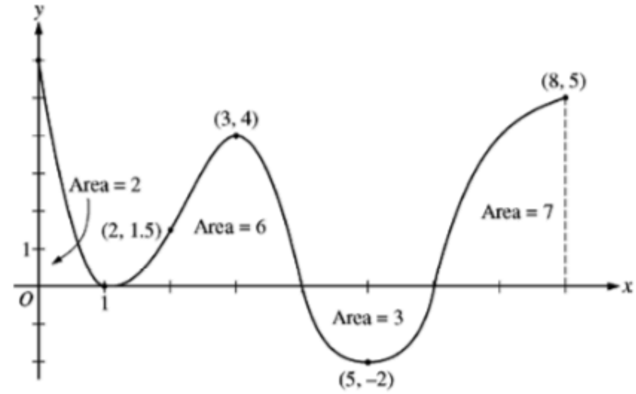
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

2013, #4



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

2019, #6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \ne 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

2014, #5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))\,g'(x)\,dx$.

2006, Form B, #6

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$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ $\left(\text{ft}/\text{sec}^2\right)$	1	5	2	1	2	4	2

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(b) Find the exact value of $\int_{0}^{30} a(t) dt$.

(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

(a)

Trapezoidal approximation for $\int_{30}^{60} |v(t)|\, dt$:

$A = \frac{1}{2}(14 + 10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185$ ft

1 : value

(b)

$\int_{0}^{30} a(t)\, dt = \int_{0}^{30} v'(t)\, dt = v(30) - v(0)$

$= -14 - (-20) = 6$ ft/sec

1 : value

(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a t in $(35, 50)$ so that $v(t) = -5$.

2 : $\begin{cases} 1 : v(35) < -5 < v(50) \\ 1 : \text{Yes; refers to IVT or hypotheses} \end{cases}$

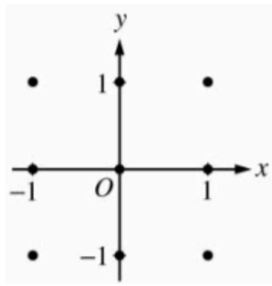
(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a t in $(0, 25)$ so that $a(t) = v'(t) = 0$.

2 : $\begin{cases} 1 : v(0) = v(25) \\ 1 : \text{Yes; refers to MVT or hypotheses} \end{cases}$

2006, Form B, #5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

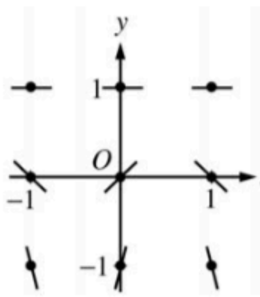
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1 : c = 1$

(c)

$$\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$$

$$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$$

$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$$

$$1 = \frac{1}{\pi}\sin(\pi) + C = C$$

$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$$

$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$

$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \quad \text{for } -\infty < x < \infty$$

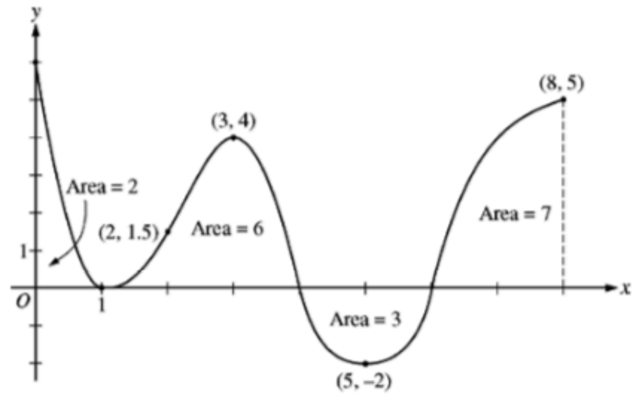
$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

2013, #4

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

1 : answer with justification

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$f(0) = f(8) + \int_8^0 f'(x)\,dx = f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$$

$$f(6) = f(8) + \int_8^6 f'(x)\,dx = f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$

$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

3 : { 1 : considers $x = 0$ and $x = 6$; 1 : answer; 1 : justification }

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

2 : { 1 : answer; 1 : explanation }

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$$

3 : { 2 : $g'(x)$; 1 : answer }

2019, #6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Solution	Scoring
(a) $h'(2) = \frac{2}{3}$	1 : answer
(b) $a'(x) = 9x^2h(x) + 3x^3h'(x)$ $a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$	3 : { 1 : form of product rule 1 : $a'(x)$ 1 : $a'(2)$
(c) Because h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$. Also, $\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3}$, so $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4$. Because $\lim_{x\to 2}(x^2 - 4) = 0$, we must also have $\lim_{x\to 2}(1-(f(x))^3) = 0$. Thus $\lim_{x\to 2} f(x) = 1$. Because f is differentiable, f is continuous, so $f(2) = \lim_{x\to 2} f(x) = 1$. Also, because f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists. Using L'Hospital's Rule, $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$ Thus $f'(2) = -\frac{1}{3}$.	4 : { 1 : $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4$ 1 : $f(2)$ 1 : L'Hospital's Rule 1 : $f'(2)$
(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$. Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x\to 2} k(x) = 4$.	1 : continuous with justification

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

2014, #5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
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(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))\,g'(x)\,dx$.

(a) $x = 1$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a relative minimum at $x = 1$.

1 : answer with justification

(b) f' is differentiable $\Rightarrow$ f' is continuous on the interval $-1 \le x \le 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$.

$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$

(c) $h'(x) = \frac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d)
$$\begin{aligned} \int_{-2}^{3} f'(g(x))g'(x)\,dx &= \left[f(g(x)) \right]_{x=-2}^{x=3} \\ &= f(g(3)) - f(g(-2)) \\ &= f(1) - f(-1) \\ &= 2 - 8 = -6 \end{aligned}$$

$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$