Sum/Difference Angle Identities

It will be helpful to have a filled in Unit Circle to refer to as you work through this lesson!

Sum Identities
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Notice how the signs between the terms differ depending on if you are using the sum or difference version of the identity.

Difference Identities
$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Example #1:

Find the exact value of cos 15°.

$$\cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

To use these identities, we need to think of two angles on the Unit Circle that we could add or subtract in order to obtain 15°. One possibility is 45° minus 30°. You also could have used 60° minus 45° or any combination of Unit Circle angles that would yield 15°. This work just illustrates one possibility. The answer would be the same regardless. Since we are subtracting, we use the Difference Identity for cosine. We use the identity, plug in the Unit Circle values, and then simplify.

Example #2:

Find the exact value of $\tan\left(\frac{11\pi}{12}\right)$**.**

$$\tan\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = \frac{\tan\frac{\pi}{4} + \tan\frac{2\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{2\pi}{3}} = \frac{1 + -\sqrt{3}}{1 - (-\sqrt{3})}$$

$$= \boxed{\frac{1-\sqrt{3}}{1+\sqrt{3}}}$$

Once we obtain radian values that sum to $\frac{11\pi}{12}$, we apply the Sum Identity for tangent, plug in the Unit Circle values, and then simplify.

You try: **Find the exact value of** $\sin\left(\frac{7\pi}{12}\right)$.

$\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$ ← option of angles to add together

$\frac{\sqrt{2}+\sqrt{6}}{4}$ ← Answer.

Example #3:

Prove the identity: $\cos\left(x+\frac{\pi}{2}\right) = -\sin x$.

$$\cos\left(x+\frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}$$
$$= \cos x \cdot (0) - \sin x\,(1)$$
$$= -\sin x \quad \checkmark$$

In these two "proofs", we start by expanding the sum or difference using the appropriate identity. Then we substitute Unit Circle Values for the radians. The simplified expression should match the desired result.

Example #4:

Prove the identity $\tan\left(\theta+\frac{\pi}{4}\right) = \frac{1+\tan\theta}{1-\tan\theta}$.

$$\tan\left(\theta+\frac{\pi}{4}\right) = \frac{\tan\theta + \tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}} = \frac{\tan\theta + 1}{1-\tan\theta} \quad \checkmark$$

You try: **Prove:** $-\cos x = \cos(x+\pi)$

Sample Answer:

$$\cos(x+\pi) = \cos x\cos\pi - \sin x\sin\pi$$
$$= \cos x\cdot(-1) - \sin x\cdot(0)$$
$$= -\cos x$$